Ageing in the contact process: Scaling behavior and universal features

Florian Baumann[1,2] **and Andrea Gambassi**[3,4]

[1]Institut für Theoretische Physik I, Universität Erlangen-Nürnberg,
Staudtstraße 7B3, D – 91058 Erlangen, Germany

[2]Laboratoire de Physique des Matériaux,‡ Université Henri Poincaré Nancy I,
B.P. 239, F – 54506 Vandœuvre lès Nancy Cedex, France

[3] Max-Planck-Institut für Metallforschung,
Heisenbergstraße 3, D–70569 Stuttgart, Germany

[4] Institut für Theoretische und Angewandte Physik, Universität Stuttgart,
Pfaffenwaldring 57, D-70569 Stuttgart, Germany

Abstract. We investigate some aspects of the ageing behavior observed in the contact process after a quench from its active phase to the critical point. In particular we discuss the scaling properties of the two-time response function and we calculate it and its *universal ratio* to the two-time correlation function up to first order in the field-theoretical ϵ-expansion ($\epsilon = 4 - d$). The scaling form of the response function does not fit the prediction of the theory of local scale invariance. Our findings are in good qualitative agreement with recent numerical results.

1. Introduction

The contact process (CP) has been introduced long ago as a simple model for the spreading of diseases [1] (see also [2, 3, 4, 5]). It is defined on a d-dimensional hypercubic lattice in which each site x is characterized by an occupation variable n_x and is either active (ill, $n_x = 1$) or inactive (healthy, $n_x = 0$). At each time step of the dynamics a site x is chosen at random. If x is inactive then it becomes active with a rate given by Λ times the fraction of its neighbouring sites which are already active (infection). Otherwise, if originally active, x becomes inactive with rate 1 (healing). In the long-time (non-equilibrium) stationary state all lattice sites will be eventually inactive (*absorbing phase*) if Λ is small enough, whereas for large enough Λ a finite fraction of sites will be active (*active phase*). These two phases are separated by a continuous phase transition for $\Lambda = \Lambda_c$ (with $\Lambda_c \simeq 3.3$ in $d = 1$, see, e.g., [4]) and a suitable order parameter is indeed provided by the average density of active sites $\langle n_x \rangle$ in the stationary state.

Extensive Monte Carlo simulations of the CP have provided a very detailed quantitative description of its dynamical behavior and of the associated non-equilibrium phase transition, which turns out to be in the same universality class as directed percolation and therefore its universal features are properly captured by the so-called *Reggeon field theory* (see, e.g., [2, 3, 6, 7]).

The directed percolation (DP) universality class has been recently the subject of renewed interest in the context of ageing phenomena [8, 9, 10, 11], which are characterized by two-time quantities (such as response and correlation functions, see below) which are homogeneous functions of their time arguments and do not display the time-translational invariance which is expected in a stationary state. The two-time quantities of interest are the connected density-density correlation function $C_{x-x'}(t, s) := \langle n_x(t) n_{x'}(s) \rangle - \langle n_x(t) \rangle \langle n_{x'}(s) \rangle$ and the response function $R_{x-x'}(t, s) := \delta \langle n_x(t) \rangle / \delta \kappa_{x'}(s)|_{\kappa=0}$ where $\kappa_{x'}(s)$ is the field conjugate to $n_{x'}(s)$ — corresponding to a local spontaneous activation rate of the lattice site x' at time s — and $\langle \ldots \rangle$ stands for the average over the stochastic realization of the process.

In [8, 9] the ageing behavior of the critical CP was investigated numerically. A system with a homogenous initial particle density (corresponding to the active phase with $\Lambda \gg \Lambda_c$) was quenched at $t = 0$ to the critical point $\Lambda = \Lambda_c$ and $R_{x=0}(t, s)$ and $C_{x=0}(t, s)$ were determined, finding the following scaling behavior ($t > s$)

$$R_{x=0}(t, s) = s^{-1-a} f_R(t/s), \qquad C_{x=0}(t, s) = s^{-b} f_C(t/s), \qquad (1)$$

where

$$f_R(y \gg 1) \sim y^{-\lambda_R/z}, \qquad f_C(y \gg 1) \sim y^{-\lambda_C/z} \tag{2}$$

and $b = 2\beta/(\nu z) = 2\delta$ [β, ν, z and $\delta := \beta/(\nu z)$ are the standard critical exponents of DP, see, e.g., [4] — the values of z and δ for $d = 1, 2$ and 3 are reported in table 1]. In particular it was found that $\lambda_C/z = 1.85(10)$ and $\lambda_R/z = 1.85(10)$ for $d = 1$ [8, 9], whereas $\lambda_C/z = 2.75(10)$ and $\lambda_R/z = 2.8(3)$ for $d = 2$ [9]. These results suggest that $\lambda_R = \lambda_C$ independently of the dimensionality $d = 1$ and 2. On the same numerical footing it was noticed — with some surprise — that $1 + a = b$, in stark contrast to the case of slow dynamics of systems with detailed balance such as Ising ferromagnets, for which $a = b$ (see, e.g., [12, 13]). The fluctuation-dissipation ratio [14, 12]

$$X(t, s) := T_b R_{x=0}(t, s)/\partial_s C_{x=0}(t, s) \tag{3}$$

and its long-time limit $X_\infty := \lim_{s\to\infty} \lim_{t\to\infty} X(t, s)$ have been used, for these magnetic systems evolving in contact with a thermal bath of temperature T_b, to detect whether the (equilibrium) stationary state has been reached, in which case the fluctuation-dissipation theorem yields $X_\infty = 1$. Given that $b = a + 1$ for the CP, the fluctuation-dissipation ratio — as defined in equation (3) — would always yield a trivial value of X_∞ and therefore it would not serve its purpose. In [8], it was suggested to consider, instead of X,

$$\Xi(t, s) := \frac{R_{x=0}(t, s)}{C_{x=0}(t, s)} = \frac{f_R(t/s)}{f_C(t/s)} \tag{4}$$

and define $\Xi_\infty := \lim_{s\to\infty} \lim_{t\to\infty} \Xi(t, s)$, which has now a finite non-trivial value $\Xi_\infty = 1.15(5)$ for the CP in $d = 1$ [8]. (We comment on the meaning of Ξ in subsection 2.2.) In addition, the numerical results of [8, 9] support the applicability of the so-called theory of local scale invariance (LSI) [15, 16, 17] to the ageing behavior of the CP. This theory tries to use *local* space-time symmetries to constrain the form of $f_R(t/s)$ and has been applied to ageing phenomena in magnetic systems quenched from an initial high-temperature state to and below the critical temperature [16, 17] and bosonic reaction-diffusion systems [18]. Recent simulations suggest that LSI holds also in the parity conserving universality class [19]. However, in the case of critical ageing of the Ising model with purely relaxational dynamics, it was shown that corrections to the prediction of LSI are present both in field-theoretical results at two loops [13, 20] and in simulations [21, 22]. A more general version of LSI [11] — the version we shall refer to in this paper — improved considerably the agreement with simulations while the disagreement with field-theoretical predictions remained. Also in the case of the

CP, numerical results indicate that the scaling function f_R predicted by LSI is incorrect for $t \simeq s$ [10, 11]. It was suggested in [11, 23] that one could possibly account for this discrepancy by extending LSI to include also the case of a nonvanishing average of the order parameter (in the present version a vanishing order parameter is implicitly assumed). As this extension is at present still lacking, in what follows we shall refer to the available version of LSI.

The previously mentioned (numerical) works leave some questions open:

(*i*) Could the relations $1 + a = b$ and $\lambda_R = \lambda_C$ have been expected?

(*ii*) Is $\lambda_R \, (= \lambda_C)$ an independent critical exponent?

(*iii*) Is Ξ_∞ a *universal* quantity like X_∞ in ageing systems with detailed balance?

(*iv*) To what extent does LSI describe some of the features of the ageing behavior in the CP?

The aim of this note is to answer these questions by adopting a field-theoretical approach.

The rest of the paper is organized as follows. In section 2 we introduce the field-theoretical model (Reggeon field theory), the formalism and we set up the general framework for the perturbative expansion. In addition we discuss the expected scaling forms for the two-time response and correlation functions, the relation among the different ageing exponents and between X and Ξ, providing complete answers to the questions (*i*), (*ii*) and (*iii*). In section 3, we calculate the response function and its long-time ratio Ξ_∞ to the correlation function up to first order in the ϵ-expansion ($\epsilon = 4 - d$) and then we compare our results to the predictions of LSI, providing an answer to question (*iv*). In section 4 we summarize our findings and present our conclusions. Some details of the calculation are reported in the appendix.

2. The field-theoretical approach

As explained in detail in [2, 3], the universal scaling properties of the CP (more generally, of the DP universality class) in the stationary state are captured by a Reggeon field-theoretical action S. In the critical case it reads

$$ S[\varphi, \tilde{\varphi}] = \int \mathrm{d}^d x \, \mathrm{d}t \, \left\{ \tilde{\varphi} \left[\partial_t - D\nabla^2 \right] \varphi - u(\tilde{\varphi} - \varphi)\tilde{\varphi}\varphi - h\,\tilde{\varphi} \right\} , \tag{5} $$

where the $\varphi(x, t)$ and $h(x, t)$ are the coarse-grained versions of the particle density $n_x(t)$ and of the spontaneous activation rate of lattice sites $\kappa_x(t)$ (external perturbation), respectively, whereas $\tilde{\varphi}(x, t)$ is the response field, $u > 0$ the bare coupling constant of the theory and D the diffusion coefficient ($D = 1$ in what follows, unless differently stated).

In terms of the action (5), the average of an observable $\mathcal{O}$ over the possible stochastic realizations of the process is given by $\langle \mathcal{O} \rangle = \int [\mathrm{d}\varphi \mathrm{d}\tilde{\varphi}] \, \mathcal{O} \, \mathrm{e}^{-S[\varphi, \tilde{\varphi}]}$. As a result, the response of $\langle \mathcal{O} \rangle$ to the external perturbation h can be computed as $\delta \langle \mathcal{O} \rangle / \delta h(x, t) = \langle \tilde{\varphi}(x, t) \mathcal{O} \rangle$, leading to the following expression for the (linear) response function $R_{x-x'}(t, s) := \delta \langle \varphi(x, t) \rangle / \delta h(x', s)|_{h=0} = \langle \varphi(x, t) \tilde{\varphi}(x', s) \rangle_{h=0}$.

The action (5) with $h = 0$ and $t \in (-\infty, \infty)$ is invariant under the duality transformation

$$\tilde{\varphi}(x, t) \xrightleftharpoons{\mathrm{RR}} -\varphi(x, -t) \tag{6}$$

(the so-called *rapidity reversal* — RR) which implies that the scaling dimensions $[\ldots]_{\mathrm{scal}}$ of the fields φ and $\tilde{\varphi}$ are equal (see, e.g, [2, 3]):

$$[\varphi(x, t)]_{\mathrm{scal}} = [\tilde{\varphi}(x, t)]_{\mathrm{scal}} = \beta/\nu . \tag{7}$$

Note that, as in the case of systems with detailed balance (DB), the scaling dimensions of the fields (with $t > 0$) do not change if RR (alternatively, DB) is broken by the presence of *initial conditions* (at time $t = 0$).

If RR is a symmetry of S, then

$$\begin{aligned} C_{x-x'}(t, s) &:= \langle \varphi(x, t) \varphi(x', s) \rangle_{h=0} - \langle \varphi(x, t) \rangle_{h=0} \langle \varphi(x', s) \rangle_{h=0} \\ &\stackrel{\mathrm{RR}}{=} \langle \tilde{\varphi}(x, -t) \tilde{\varphi}(x', -s) \rangle_{h=0} - \langle \tilde{\varphi}(x, -t) \rangle_{h=0} \langle \tilde{\varphi}(x', -s) \rangle_{h=0} = 0 , \end{aligned} \tag{8}$$

where the last equality is a consequence of causality [24], and therefore correlations vanish in the stationary state of the CP. On the other hand, a suitable initial condition (say, at time $t = 0$) can effectively generate correlations which decay for $t > 0$. Generally speaking one expects this decay to be exponential in t for $\Lambda \neq \Lambda_c$, due to a finite relaxation time both in the absorbing and active phase, whereas an algebraic decay is expected in the critical case $\Lambda = \Lambda_c$ we are interested in. (This picture is indeed confirmed by the numerical results of [8].) Accordingly, $C_{x=0}(t, s) := \langle \varphi(x, t) \varphi(x, s) \rangle - \langle \varphi(x, t) \rangle \langle \varphi(x, s) \rangle$ does no longer vanish during the relaxation from the initial condition and for $t, s \neq 0$ it has the same scaling dimension as $R_{x=0}(t, s)$. Two straightforward consequences of this fact are:

(A) $1 + a = b$ [see equation (1)], which could have been expected on this basis and therefore is valid beyond the mere numerical coincidence† and

(B) $\Xi(t, s)$ [see equation (4)] — and therefore Ξ_∞ — is a ratio of two quantities with the same engineering and scaling dimensions and therefore it is a *universal* function (see, e.g., [26]) which takes the same value in all the models belonging to the same universality class (in particular, the lattice CP, the DP and the model (5) on the continuum). This is analogous to the case of X_∞ in systems with DB.

---

† The same conclusion can be drawn from the analysis presented in [25] (see also [2, 3])

2.1. Scaling forms

The non-equilibrium dynamics of the CP after a sudden quench (at time $t = 0$) from the active phase $\Lambda \gg \Lambda_c$ (characterized by $\langle n_x(t < 0) \rangle = 1$ on the lattice and $\langle \varphi(x, t < 0) \rangle \neq 0$ on the continuum) to the critical point $\Lambda = \Lambda_c$ is partly analogous to the non-equilibrium relaxation of Ising systems with dissipative dynamics after a quench from a magnetized state (e.g., a low-temperature one with $T \ll T_c$, where T_c is the critical temperature) to the critical point $T = T_c$ [27]. In both cases the order parameter $m(t) := \langle \varphi(x, t) \rangle$‡ § provides a background for the fluctuations, with a *universal* scaling behavior [28, 29]

$$m(t) = A_m m_0 t^{\theta + \hat{a}} \mathcal{F}_M(B_m m_0 t^\varsigma) \qquad (9)$$

where $m_0 := \langle \varphi(x, t = 0) \rangle$ is the initial value of the order parameter, $\hat{a}$ is related to the scaling dimensions of the fields φ and $\tilde{\varphi}$ in real space via $\hat{a} = (d - [\varphi]_{\text{scal}} - [\tilde{\varphi}]_{\text{scal}})/z$ (we recall that $[\varphi]_{\text{scal}} = \beta/\nu$), θ is the so-called initial-slip exponent [29] and $\varsigma = \theta + \hat{a} + \beta/(\nu z)$. In equation (9), $\mathcal{F}_M(v)$ is a universal scaling function once the non-universal amplitudes A_m and B_m have been fixed by suitable normalization conditions, e.g., $\mathcal{F}_M(0) = 1$ and $\mathcal{F}_M(v \to \infty) = v^{-1} + O(v^{-2})$. [Note that $\mathcal{F}_M(v \to \infty) \sim v^{-1}$ is required in order to recover the well-know long-time decay of the order parameter $m(t) \sim t^{-\beta/(\nu z)}$.] Within the Ising universality class $\hat{a} = (2 - \eta - z)/z$ and $\theta \neq 0$ [29, 13] whereas within the DP class $\hat{a} = -\eta/z$ (indeed $[\varphi]_{\text{scal}} = [\tilde{\varphi}]_{\text{scal}} = (d + \eta)/2$ [2]) and $\theta = 0$ [28]. In both cases it turns out that the width $\Delta_0 = \langle [\varphi(x, 0) - m_0]^2 \rangle$ of the initial distribution of the order parameter φ — assumed with short-ranged spatial correlations — is irrelevant [29, 28] and controls only corrections to the leading scaling behavior, so that an initial state with $\Delta_0 \neq 0$ is asymptotically equivalent to one with no fluctuations $\Delta_0 = 0$. Equation (9) clearly shows that a non-vanishing value of the initial order parameter introduces an additional time scale in the problem $\tau_m = (B_m m_0)^{-1/\varsigma}$ [2, 28, 27]. The long-time limit we are interested in is characterized by times $\gg \tau_m$ and therefore the relevant scaling properties can be formally explored in the limit $m_0 \to \infty$, i.e., $\tau_m \to 0$.

According to the analogy discussed above one expects the following scaling form for the two-time response function in momentum space (see, e.g., [2, 28] and section 3 in [27])

$$R_{q=0}(t > s, s) = A_R (t - s)^{\hat{a}} (t/s)^\theta F_R(s/t, B_m m_0^{1/\varsigma} t) , \qquad (10)$$

‡ For magnetic systems φ represents the local magnetization whereas in the present case $\langle \varphi \rangle$ is the coarse-grained density of active sites.
§ Hereafter we assume invariance of the system under space translations.

where F_R is a universal scaling function once the non-universal amplitude A_R has been fixed by requiring, e.g., $F_R(0,0) = 1$. In the case of the DP universality class, the two-time Gaussian correlation function $C_x^{(0)}(t,s)$ vanishes for $m_0 = 0$ — apart from irrelevant corrections due to a finite Δ_0 — whereas the two-time Gaussian response function $R_x^{(0)}(t,s)$ is invariant under time translations. Taking into account causality, it is easy to realize that all the diagrammatic contributions to the full response function (with $u \neq 0$) have the same invariance and therefore $F_R(s/t, 0) = 1$. (As opposed to the case of the Ising universality class, where $\theta \neq 0$ and $F_R(s/t, 0) \neq 1$ [30, 20, 13, 21].) In fact, the case $m_0 = 0$ would correspond on the lattice to an initially empty system with $\Lambda = \Lambda_c$ where nothing happens as long as one does not switch on the external field $\kappa_{x'}(s)$. Accordingly, the response to $\kappa_{x'}(s)$, at time $s + \Delta t$ has to depend only on Δt.

In the long-time limit $t > s \gg \tau_m$ one expects F_R to become independent of the actual value of m_0 and therefore

$$F_R(x, v \to \infty) = \frac{\mathcal{A}_R}{A_R} x^{\theta - \bar{\theta}} \mathcal{F}_R(x), \tag{11}$$

where $\bar{\theta}$ is an additional exponent [related to λ_R, cf equation (18)], so that the resulting scaling is

$$R_{q=0}(t > s, s) = \mathcal{A}_R (t-s)^{\hat{a}} (t/s)^{\bar{\theta}} \mathcal{F}_R(s/t) . \tag{12}$$

$\mathcal{A}_R$ is a *non-universal* constant which can be fixed by requiring $\mathcal{F}_R(0) = 1$ and which has a *universal* ratio to A_R [see equation (11)]. For the Ising universality class, is was found [27]

$$\bar{\theta} = -\left(1 + \hat{a} + \frac{\beta}{\nu z}\right). \tag{13}$$

On the other hand, it is easy to see that the scaling arguments which were used to drawn this conclusion for the response function (see section 3 in [27]) are valid also for the CP.

In [25] the scaling behavior of the two-time correlation function $C_{q=0}(t,s)$, after a quench from the active phase to the critical point, has been discussed with the result that (see equations (3.3) and (4.14) in [25])

$$C_{q=0}(t > s, s) = \mathcal{A}_C (t-s)^{\hat{a}} (t/s)^{\bar{\theta}} \mathcal{F}_C(s/t) \tag{14}$$

where the non-universal amplitude $\mathcal{A}_C$ can be fixed by requiring $\mathcal{F}_C(0) = 1$, and $\bar{\theta}$ is indeed given by equation (13). Note that in analogy with Ξ [see equation (4)] one can

also define (we assume $t > s \gg \tau_m$)

$$\widehat{\Xi}(t,s) := \frac{R_{q=0}(t,s)}{C_{q=0}(t,s)} = \frac{\mathcal{A}_R \, \mathcal{F}_R(s/t)}{\mathcal{A}_C \, \mathcal{F}_C(s/t)} \tag{15}$$

[where we have used the scaling forms (12) and (14)] and

$$\widehat{\Xi}_\infty := \lim_{s\to\infty} \lim_{t\to\infty} \widehat{\Xi}(t,s) = \frac{\mathcal{A}_R}{\mathcal{A}_C} \tag{16}$$

which are, as $\Xi(t,s)$ and Ξ_∞, a *universal* function and amplitude ratio, respectively. Although in general $\Xi(t,s) \neq \widehat{\Xi}(t,s)$, the argument discussed in [30, 13] can be used to conclude that $\Xi_\infty = \widehat{\Xi}_\infty$ also in this case.

In what follows we focus on $R_{q=0}$ and $C_{q=0}$, i.e., the response and correlation function of the spatial average of the density of active sites ($N^{-1}\langle \sum_x n_x \rangle$ on a lattice with N sites). On the other hand, the corresponding scaling forms equation (12) and (14) can be easily generalized to $q \neq 0$ taking into account that this amounts to the introduction of an additional scaling variable $y = A_D D q^z (t-s)$ where A_D is a dimensional non-universal constant which can be fixed via a suitable normalization condition. Accordingly, the scaling behavior of the autoresponse $R_{x=0}$ and autocorrelation $C_{x=0}$ functions can be easily worked out from equations (12) and (14), leading to the identification of the exponents a and $\lambda_{R,C}$ in equations (1) and (2) as

$$1 + a = b = \frac{d}{z} - \hat{a} = \frac{2\beta}{\nu z} = 2\delta \tag{17}$$

and

$$\frac{\lambda_C}{z} = \frac{\lambda_R}{z} = \frac{d}{z} - \hat{a} - \bar{\theta} = 1 + \delta + \frac{d}{z}. \tag{18}$$

Therefore we conclude that:

(C) λ_R is *equal* to λ_C beyond the mere numerical coincidence observed in [8, 9] and

(D) $\lambda_R = \lambda_C$ is *not* an independent critical exponent but it is given by

$$\lambda_R = z + z\delta + d. \tag{19}$$

In table 1 we compare the values of $\lambda_{R,C}$ obtained from this scaling relation and the available estimates of δ and z to the results of fitting the asymptotic behavior of the scaling functions $f_{C,R}$ [see equations (1) and (2)] from numerical data [8, 9]. The results reported in the last three columns are in quite good agreement both for $d = 1$ and 2.

d	δ	z	$1+\delta+d/z$	λ_R/z	λ_C/z
1	0.159464(6)	1.580745(10)	1.792077(10)	1.85(10) [8] 1.9(1) [9] 1.76(5) [10]	1.85(10) [8] 1.9(1) [9]
2	0.451	1.76(3)	2.58(2)	2.75(10) [9]	2.8(3) [9]
3	0.73	1.90(1)	3.30(1)	not available	

Table 1. Comparison between the direct numerical estimates of $\lambda_{R,C}/z$ and the predictions of the scaling relation $\lambda_R/z(=\lambda_C/z)=1+\delta+d/z$ [see equation (18)] in which the available estimates of δ and z (taken from table 2 in [4]) are used. The values of $\lambda_{R,C}/z$ reported in the last two columns have been obtained by fitting the scaling behavior of the autoresponse and autocorrelation function determined via density-matrix renormalization-group computation [8] and by Monte Carlo simulations [9, 10] of the contact process.

The conclusions (A–D) we have drawn so far provide complete answers to the questions (*i–iii*) which we have posed at the end of the Introduction.

2.2. Relation between X and Ξ

In [8] Ξ has been introduced on the basis of a formal analogy with the fluctuation-dissipation ratio $X(t,s)$. Here we argue that indeed X and Ξ play the same role in different circumstances.

The stationary state of a system with detailed balance is characterized by the time-reversal (TR) symmetry of the dynamics, which qualifies the state as an *equilibrium* one with a certain temperature T. Time-translational invariance (TT) and TR symmetry of the dynamics in the equilibrium state imply the fluctuation-dissipation theorem $TR_{x,q}(t,s)=\partial_s C_{x,q}(t,s)$, leading to $X=1$ [see equation (3)]. In this sense $X\neq 1$ is a signature of the fact that the system has not reached its stationary (equilibrium) state. (This is usually due to slowly relaxing modes which prevent the system from "forgetting" the initial conditions of the dynamics [31, 13].) When detailed balance does not hold, TR is no longer a symmetry of the stationary state. In the specific case of the DP universality class, however, the *non-equilibrium* stationary state is characterized by a different symmetry, the rapidity-reversal [RR, see equation (6)], which leads — as discussed in subsection 2.1, equation (8) — to vanishing correlations and therefore to $\Xi^{-1},\widehat{\Xi}^{-1}=0$. Accordingly, $\Xi^{-1},\widehat{\Xi}^{-1}\neq 0$ provides a signature of the fact that the system is not in its stationary state. In this sense Ξ ($\widehat{\Xi}$) is analogous to X for both of them indicate if the stationary state is eventually reached. In particular this occurs *generically* after a perturbation (e.g., a sudden change in the temperature T or the spreading rate Λ) and therefore $X_\infty=1$ or $\Xi_\infty^{-1}=\widehat{\Xi}_\infty^{-1}=0$. On the other hand, it may happen that for

particular choices of the external parameters (e.g., T or Λ), slow modes emerge which prevent the system from achieving its stationary state. As a consequence, during this neverending relaxation of the system, TT is broken together with the possible additional symmetries which characterize the stationary state.‖ This is what happens when ageing takes place in critical systems with detailed balance. In the next subsection we explicitly show that this is also the case for the contact process.

2.3. The Gaussian approximation

The analytic computation of the response function follows the same steps as those discussed in [27] (see also [25]) for the non-equilibrium relaxation of the Ising model relaxing from a state with non-vanishing value of the magnetization. Here we briefly outline the calculation. We assume that the initial state has a non-vanishing order parameter $m_0 = \langle \varphi(x, t = 0) \rangle$ which is spatially homogeneous, so that the ensuing relaxation is characterized by translational invariance in space. It is convenient to subtract from the order parameter field $\varphi(x,t)$ its average value $m(t) = \langle \varphi(x,t) \rangle$, by introducing

$$\psi(x,t) := \varphi(x,t) - m(t) \quad \text{and} \quad \tilde{\psi}(x,t) := \tilde{\varphi}(x,t) \,, \tag{20}$$

leading to $\langle \psi(x,t) \rangle = 0$ during the relaxation. The action S [see equation (5)] becomes, in terms of these fields,

$$S[\psi, \tilde{\psi}] = \int \mathrm{d}^d x \mathrm{d}t \left\{ \tilde{\psi} \left[\partial_t - \nabla^2 + 2\sigma \right] \psi - \sigma \tilde{\psi}^2 - u(\tilde{\psi} - \psi)\tilde{\psi}\psi \right\} \,, \tag{21}$$

where $\sigma(t) := u\, m(t)$ satisfies the equation of motion (no-tadpole condition)

$$\partial_t \sigma + \sigma^2 + u^2 \langle \psi^2 \rangle = 0. \tag{22}$$

Equations (22) and (21) are the starting point for our calculations. The Gaussian response and correlation function are obtained by neglecting anharmonic terms in equation (21):

$$R_q^{(0)}(t,s) = \theta(t-s) \exp \left\{ -q^2(t-s) - 2 \int_s^t \mathrm{d}t' \sigma(t') \right\} \quad \text{and} \tag{23}$$

$$C_q^{(0)}(t,s) = 2 \int_0^{t_<} \mathrm{d}t' \, R_q^{(0)}(t,t')\sigma(t')R_q^{(0)}(s,t') \tag{24}$$

‖ The one described here is the typical pattern of the spontaneous breaking of a symmetry. See, e.g., [31].

where $t_< := \min\{t, s\}$. [In what follows we denote the results of the Gaussian approximation with the superscript (0).] In particular, solving (22) with $u = 0$ one finds (see also [25] and the appendix of [9]) $\sigma^{(0)}(t) = (t + \sigma_0^{-1})^{-1}$, which scales according to (9) ($\beta^{(0)} = 1$, $\nu^{(0)} = 1/2$, $z^{(0)} = 2$ and $\eta^{(0)} = 0$, see, e.g., [24]) with $A_m^{(0)} = 1$, $B_m^{(0)} = u$ and $\mathcal{F}_M^{(0)}(x) = (1 + x)^{-1}$. As explained in section 2.1, the leading behavior for $t > s \gg \tau_m = (B_m m_0)^{-1/\varsigma} = \sigma_0^{-1}$ can be explored by taking the limit $m_0 \propto \sigma_0 \to \infty$ from the very beginning. Accordingly, $\sigma^{(0)}(t) = t^{-1}$ and

$$R_q^{(0)}(t, s) = \theta(t - s) \left(\frac{s}{t}\right)^2 e^{-q^2(t-s)}, \tag{25}$$

$$C_q^{(0)}(t, s) = 2 e^{-q^2(t+s)} (t\,s)^{-2} \int_0^{t_<} \mathrm{d}t'\, t'^3 e^{2q^2 t'}, \tag{26}$$

which (for $q = 0$) display the expected scaling behaviors (12) and (14) with $\mathcal{A}_R^{(0)} = 1$, $\mathcal{A}_C^{(0)} = 1/2$ and $\mathcal{F}_{C,R}^{(0)}(x) \equiv 1$. Equations (15) and (16) yield $\widehat{\Xi}^{(0)}(t, s) = \widehat{\Xi}_\infty^{(0)} = 2$, which is exact for $d > 4$ but it is almost twice as large as the result which was found in $d = 1$, $\widehat{\Xi}_\infty = 1.15(5)$ [8]. As explained in subsection 2.1, $\Xi^{-1} \neq 0$ is a signal of the breaking of the RR symmetry characterizing the non-equilibrium stationary state of the CP (and of the DP universality class in general). Here we show explicitly (within the Gaussian approximation — though the conclusion is expected to be valid also beyond the approximation) that only the homogeneous fluctuation mode $q = 0$ is, at criticality, responsible for such a breaking, as in the case of systems with detailed balance (see, e.g, [13, 20, 27, 30]). Indeed, let us generalize equation (15) to modes with $q \neq 0$:

$$\widehat{\Xi}_q^{-1}(t > s, s) := \frac{C_q(t, s)}{R_q(t, s)}. \tag{27}$$

Within the Gaussian approximation one readily finds [see equations (25) and (26)]

$$\left[\widehat{\Xi}_q^{(0)}(t, s)\right]^{-1} = \frac{1}{2} \frac{4!}{y^4} \left(e^{-y} - 1 + y - \frac{y^2}{2!} + \frac{y^3}{3!}\right)\bigg|_{y=2Dq^2 s}, \tag{28}$$

where the additional scaling variable $y = A_D D q^z s = D q^2 s$ appears. This expression — as the Gaussian fluctuation-dissipation ratio for systems with detailed balance (see, e.g, [30]) — is independent of t and depends on y only. In particular, in the long-time limit $s \gg q^{-2}$ one has $\left[\widehat{\Xi}_q^{(0)}(y \gg 1)\right]^{-1} \sim 1/y \to 0$, for any mode with $q \neq 0$, indicating that the RR symmetry is asymptotically realized. On the other hand, for $q = 0$, $\left[\widehat{\Xi}_q^{(0)}(y = 0)\right]^{-1} = 1/2$, independently of s. For a quench into the active phase $\Lambda > \Lambda_c$ the action S and the propagators (25) and (26) get modified according to $q^2 \to q^2 + r$, with $r > 0$ (see, e.g., [3, 2]). Therefore in this case, $y = 2D(q^2 + r)s$, yielding $\left[\widehat{\Xi}_q^{(0)}(y \gg 1)\right]^{-1} \to 0$ for $s \gg r^{-1}$ and independently of q, in agreement with what has been stated in subsection 2.2 and with the available numerical evidences [8].
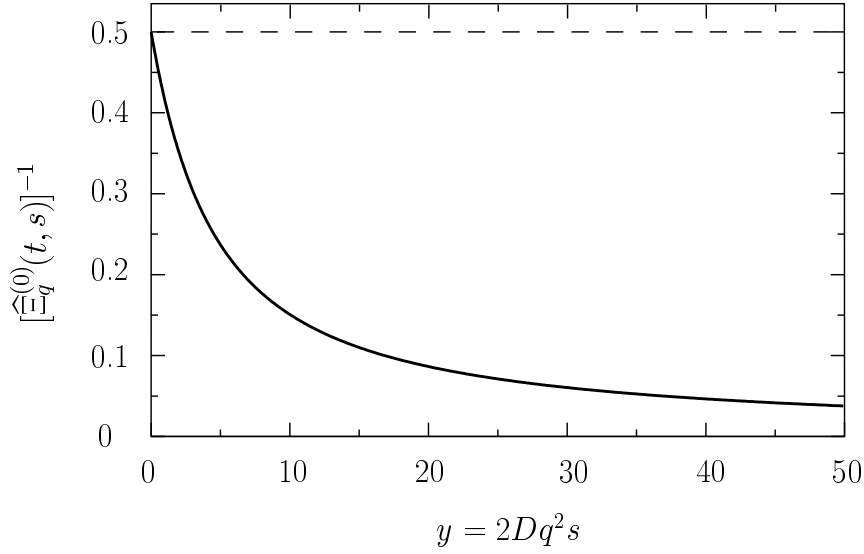
Figure 1. Ratio $[\widehat{\Xi}_q^{(0)}(t,s)]^{-1}$ of the correlation function to the response function [see equation (27)] within the Gaussian approximation (which becomes exact for $d > 4$). In the long-time limit $s \to \infty$, $[\widehat{\Xi}_q^{(0)}(t,s)]^{-1}$ vanishes, unless $q = 0$, in which case it takes the value $1/2$. $[\widehat{\Xi}_q^{(0)}]^{-1} \neq 0$ signals the breaking of RR [see equation (6) and subsection 2.2].

3. The response function

In this subsection we determine the response function up to first order in $\epsilon = 4 - d$ (4 being the upper critical dimension d_{ucd} of the model [3], above which the Gaussian results become exact). Some of the details of the calculation are provided in the appendix.

For future reference we recall that the critical exponents of the DP universality class are [2, 4, 3] $\eta = -\epsilon/6 + O(\epsilon^2)$, $z = 2 - \epsilon/12 + O(\epsilon^2)$, $\beta/\nu = (d + \eta)/2$ and therefore [see equation (13)]

$$\hat{a} = -\frac{\eta}{z} = \frac{\epsilon}{12} + O(\epsilon^2) \quad \text{and} \quad \bar{\theta} = -2 + \frac{\epsilon}{6} + O(\epsilon^2). \tag{29}$$

The expression for the renormalized response function is (where $x := s/t \leq 1$)

$$R_{R,q=0}(t,s) = x^2 \left\{ 1 + \epsilon \left[-\frac{1}{4} \ln x + \frac{1}{12} \ln s \right. \right.$$
$$\left. \left. + \frac{\pi^2}{12} + \frac{\ln(1-x)}{x} - \frac{1}{2} \text{Li}_2(x) - \frac{11}{12} \ln(1-x) + \frac{x}{12} \right] \right\} + O(\epsilon^2), \tag{30}$$

(see the appendix for details) which can be cast in the expected scaling form (12) with

$$\mathcal{A}_R = 1 - \epsilon \left(1 - \frac{\pi^2}{12} \right) + O(\epsilon^2), \tag{31}$$

$$\mathcal{F}_R(x) = 1 + \epsilon \left[1 + \frac{x}{12} + \left(\frac{1}{x} - 1 \right) \ln(1 - x) - \frac{1}{2} \text{Li}_2(x) \right] + O(\epsilon^2), \qquad (32)$$

[note that $\mathcal{F}_R(0) = 1$ and $\mathcal{F}_R(x \to 1^-) = 1 + \epsilon(13 - \pi^2)/12 + O(\epsilon^2)$] and the proper exponents (29).

3.1. Comparison with LSI

For the response function, LSI provides the following prediction [11]:

$$R_{q=0}(t > s, s) = r_0 s^A x^B (1 - x)^C, \qquad (33)$$

where $x := s/t$ and A, B, C and r_0 are free parameters. In a first version of LSI it was assumed $A = C$ but more recently it has been suggested that this constraint can be relaxed [17, 11]. On the other hand, requiring $R_{q=0}$ to have the correct scaling dimension (see subsection 2.1), leads to $A = \hat{a} = -2\delta + d/z$. In addition, the well-established short-time behavior $s/t \to 0$, from equation (12), leads to $B = -\bar{\theta} - \hat{a} = 1 + \delta$. The last free parameter, C, can be in principle fixed by requiring the response function to display a *quasi-stationary* regime (analogous to the quasi-equilibrium regime in systems with detailed balance, see, e.g., Section 2.5 in [13]) for $\Delta t \ll s$ where $\Delta t := t - s$, in which the behavior of the system depends only on Δt and not on s. This would require $C = A$ [10]. In turn, comparing with equation (12), this would imply $\mathcal{F}_R(x) = 1, \forall x \in [0, 1]$ which is in contrast with the analytical expression in equation (32). The discrepancy between the actual scaling behavior of the CP and the prediction of LSI with $A = C$, was originally overlooked in [8] but then it became apparent for $t/s \lesssim 2$ in the detailed numerical analysis carried out in [10], which is also supported by our analytical result. On the other hand, if one questions the existence of such a quasi-stationary regime in the CP, C remains an available fitting parameter. By adjusting it the agreement between numerical data and LSI improves significantly [11], down to $t/s - 1 \simeq 10^{-1}$, being discrepancies observed only for smaller $t/s - 1$ [10]. It has been suggested [11] that such discrepancies might be caused by corrections to scaling due to having $t - s$ of the order of microscopic time scales when exploring the regime $t/s - 1 \simeq 0$ with finite s — However, this does not seem to be the case for the data reported in [10].

In order to compare the analytic expression of $R_{q=0}(t, s)$ to the prediction of LSI with $C \neq A$ we introduce the quantity $\mathcal{R}(x) := \mathcal{A}_R^{-1}(t - s)^{-\hat{a}}(t/s)^{-\bar{\theta}} R_{q=0}(t, s)$ where $\mathcal{A}_R$ is fixed by the requirement $\mathcal{R}(x \to 0) = 1$. Accordingly, the field-theoretical prediction (12), yields (for $d < 4$)

$$\mathcal{R}_{\text{FT}}(x) = \mathcal{F}_R(x) = 1 + \epsilon \, \mathfrak{f}(x) + O(\epsilon^2) \qquad (34)$$

where $\mathfrak{f}(x)$ is given by the expression in square brackets in equation (32):

$$\mathfrak{f}(x) := 1 + \frac{x}{12} + \left(\frac{1}{x} - 1\right)\ln(1-x) - \frac{1}{2}\text{Li}_2(x), \tag{35}$$

with $\mathfrak{f}(1) = (13 - \pi^2)/12$. LSI predicts, instead,

$$\mathcal{R}_{\text{LSI}}(x) = (1-x)^{\Delta C} \tag{36}$$

where $\Delta C := C - A$. Above the upper critical dimension $d_{\text{ucd}} = 4$, $\mathcal{R}_{\text{FT}}(x) \equiv 1$ and $\mathcal{R}_{\text{LSI}}$ fit it with $\Delta C = 0$. Assuming continuity of critical exponents, one expects for $d < 4$ (i.e., $\epsilon > 0$), $\Delta C = c_1 \epsilon + O(\epsilon^2)$, — where c_1 is a fitting coefficient — and therefore $\mathcal{R}_{\text{LSI}}(x) = 1 + \epsilon c_1 \ln(1-x) + O(\epsilon^2)$. Of course $\mathcal{R}_{\text{FT}} - \mathcal{R}_{\text{LSI}} = \epsilon[\mathfrak{f}(x) - c_1 \ln(1-x)] + O(\epsilon) \neq 0$ for $\epsilon > 0$ (i.e., $d < 4$) whatever the choice of c_1 is, as it is clear by inspecting the expression (35). Strictly speaking LSI in its present form *is not* a symmetry of $R_{q=0}(t,s)$, whatever ΔC is, at least sufficiently close to d_{ucd} (small ϵ). Nonetheless, the shape of $\mathfrak{f}(x)$ is such that a proper choice of c_1 can reduce the difference $\mathcal{R}_{\text{FT}} - \mathcal{R}_{\text{LSI}}$. In figure 2 we report the comparison between $\ln \mathcal{R}_{\text{FT}}(x) = \epsilon \mathfrak{f}(x) + O(\epsilon^2)$ and $\ln \mathcal{R}_{\text{LSI}}(x) = \epsilon c_1 \ln(1-x) + O(\epsilon^2)$ with $c_1 = 0$ (i.e., $C = A$) and $c_1 = -0.081(2)$ (corresponding to $C \neq A$). This plot
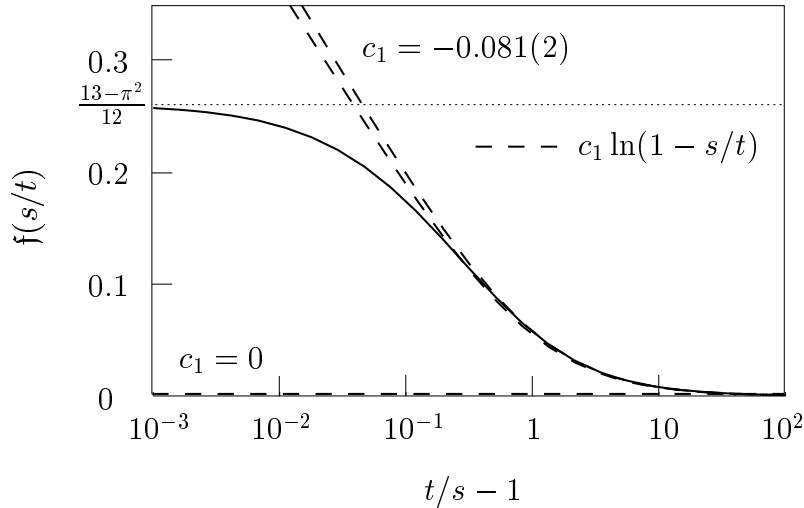
Figure 2. Comparison between $\epsilon^{-1} \ln \mathcal{R}_{\text{FT}}(x)$ and $\epsilon^{-1} \ln \mathcal{R}_{\text{LSI}}(x)$ for two different values of the fit-parameter c_1.

clearly indicates that the scaling function $\mathcal{R}_{\text{FT}}(x)$ shows corrections to the behavior predicted by LSI with $C = A$ already for $t/s - 1 \lesssim 8$, i.e., $0.11 \lesssim x \leq 1$, for which $\mathcal{R}_{\text{FT}}(x)/\mathcal{R}_{\text{LSI}}(x) \gtrsim 1 + 0.01\epsilon + O(\epsilon^2)$. On the other hand the same discrepancy is observed only for $t/s - 1 \lesssim 0.2[0.13]$, i.e., $0.83[0.88] \lesssim x \leq 1$ for $c_1 = -0.083[-0.079]$ and therefore a suitable choice of c_1 (i.e., C) reduces by more than one order of magnitude the value

of $t/s - 1$ below which the difference between the predicted scaling behavior and LSI becomes sizable. This is analogous to what has been observed in numerical data [10, 32] for the scaling of the autoresponse function $R_{x=0}(t, s)$. (See, e.g., figure 2 in [10].) On the other hand, the field-theoretical results presented here are free from numerical artefacts and corrections to scaling, which were invoked in [11] to explain the findings of [10].

As a result of our fitting procedure, we have determined the "ϵ-expansion" of the exponent ΔC, finding $\Delta C = -0.081(2)\epsilon + O(\epsilon^2)$. According to the notations adopted in [11] $\Delta C = -(a' - a)$ and therefore we conclude that $a' - a = +0.081(2)\epsilon + O(\epsilon^2)$. For the one-dimensional contact process it was found $a' - a = +0.270(10)$ [11, 10]¶, whose sign is indeed in agreement with our result (generally speaking, the sign of the correction to the Gaussian behavior is correctly captured by the lowest non-trivial order in the standard ϵ-expansion). Insisting in interpreting our result as an ϵ-expansion we can even attempt an estimate of $a' - a$ in physical dimensions $\epsilon = 1$, 2 and 3. It comes as a surprise that not only the order of magnitude is the correct one but also the estimate for the one-dimensional CP (i.e., $\epsilon = 3$) $a' - a \simeq +0.24$ is in very good agreement with the actual numerical result.

In passing, we note that a mechanism similar to what has been described here could possibly explain the (apparent) agreement between LSI with $C \neq A$ and the scaling form of the integrated response function of the magnetization in the Ising model quenched from high temperatures to the critical point [21, 11].

3.2. One-loop prediction for Ξ

The correlation function was already calculated in [25], up to one loop in the ϵ-expansion, with the result

$$2\mathcal{A}_C = A \times \left[1 - \frac{\epsilon}{6}F(\infty)\right] + O(\epsilon^2) = 1 + \frac{\epsilon}{6}\left(\frac{9\pi^2}{20} - \frac{361}{80}\right) + O(\epsilon^2) \quad (37)$$

and

$$\mathcal{F}_C(x) = 1 - \frac{\epsilon}{6}\left[F(x^{-1}) - F(\infty)\right] + O(\epsilon^2) \quad (38)$$

¶ Note that in [10] the value of a' is reported with the wrong sign.

where A and F are given, respectively, in equation (4.4) and (3.5) therein. Accordingly, we obtain for the ratio $\widehat{\Xi}_\infty$ in (16) the value

$$\widehat{\Xi}_\infty = 2\left[1 - \epsilon\left(\frac{119}{480} - \frac{\pi^2}{120}\right)\right] + O(\epsilon^2) . \tag{39}$$

Clearly the Gaussian value of $\widehat{\Xi}_\infty$ gets a downward correction ($119/480 - \pi^2/120 \simeq 0.166$) for $\epsilon > 0$. This is qualitatively satisfying, as it brings the theoretical estimate closer to the value which was determined numerically for $d = 1$ [$\Xi_\infty = 1.15(5)$]. In addition, a direct extrapolation of equation (39) to $\epsilon = 3$ (neglecting possible $O(\epsilon^2)$ terms) yields $\widehat{\Xi}_\infty^{d=1} = 1.01$ which is — for a one-loop calculation — in a surprisingly good agreement with the actual value.[+] More reliable theoretical estimates can only be based on the knowledge of higher-loop contributions.

4. Conclusions

In this note we have studied the ageing behavior in the contact process by adopting a field-theoretical approach. The results presented here equally apply to all the models belonging to the same universality class as the contact process, such as the directed percolation.

We confirmed analytically the scaling behavior

$$R_{x=0}(t, s) = s^{-1-a} f_R(t/s), \qquad C_{x=0}(t, s) = s^{-b} f_C(t/s), \tag{40}$$

which was observed numerically for the two-time response and correlation functions [8, 9] after a quench from the active phase to the critical point. In addition we have shown that:

(a) The relation $b = 1 + a$, due to the rapidity-reversal symmetry, and $\lambda_R = \lambda_C$ [see equation (2)], hold beyond the numerical evidences provided in [8, 9].

(b) $\lambda_{R,C}$ are related to known static and dynamic exponents via $\lambda_{R,C} = z(1+\delta)+d$ [see equation (19)], which is confirmed by the available numerical data (see table 1).

(c) The ratio Ξ of the response to the correlation function [equation (4)] defines a *universal function* and its long-time limit Ξ_∞ a *universal amplitude ratio*, which can therefore be studied within a field-theoretical approach.
    The result of the ϵ-expansion around $d = 4$ [see equation (39)] is in qualitative agreement with the numerical estimate of $\Xi_\infty^{d=1}$ [8].

[+] Extrapolating the result (39) as $\widehat{\Xi}_\infty^{(\text{extr})} = 2/[1 + \epsilon(119/480 - \pi^2/120)]$ would give 1.33 for $\epsilon = 3$, still not too far from the numerical result.

(d) After a quench, $\Xi_\infty^{-1} \neq 0$ for the CP, like $X_\infty - 1 \neq 0$ for systems satisfying detailed balance, is a signature of the spontaneous breaking of the symmetry $\mathfrak{S}$ associated with the corresponding non-equilibrium ($\mathfrak{S}$ = rapidity-reversal) and equilibrium ($\mathfrak{S}$ = time-reversal) stationary states.

(e) The *universal* scaling function $\mathcal{F}_R(x)$ — describing the ageing behavior of the response function $R_{q=0}(t, s)$ [see equation (12)] — *does not* agree with the prediction of the theory of local scale invariance when the non-Gaussian fluctuations are taken into account [specifically already at $O(\epsilon)$ for $\epsilon > 0$, see equation (32)]. This provides an additional example against the general applicability of the present form of LSI to critical systems. It remains to be seen whether an extension of LSI to systems with nonvanishing order parameter will be able to describe correctly the scaling function of the CP.

## Appendix A. Diagrammatic expansion of the response function

In this appendix we provide some details of the calculation of the response function. First of all we solve equation (22) — to first order in the effective coupling constant $\tilde{v} := (8\pi)^{-d/2} u^2/D^2$ — where $\langle \psi^2 \rangle$ is given by

$$\langle \psi^2(x,t) \rangle = \int (\mathrm{d}q) C_q^{(0)}(t,t) + O(\tilde{v}) = (8\pi)^{-\frac{d}{2}} r_d t^{-\frac{d}{2}} + O(\tilde{v}) . \tag{A.1}$$

For later convenience we introduce the notation $(\mathrm{d}q) := \mathrm{d}^d q/(2\pi)^d$ [and dimensional regularisation of the integrals is understood] and $r_d = 12\Gamma(1 - d/2)/\Gamma(5 - d/2) = -12/\epsilon + 3 + O(\epsilon)$. Accordingly, $\sigma(t)$ is given by

$$\sigma(t) = \sigma^{(0)}(t) \left(1 - \tilde{v} \frac{r_d}{\epsilon/2 + 1} t^{\epsilon/2} \right) + O(\tilde{v}^2) , \tag{A.2}$$

where $\sigma^{(0)}(t) = 1/t$ is the solution within the Gaussian approximation. (As we are interested in the long-time limit, we have taken $\sigma_0 \to \infty$.) As expected, the
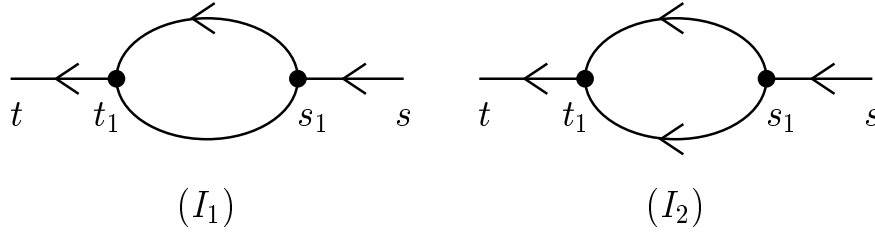
Figure A1. Diagrammatic contributions to the response function up to first order in v. Directed and undirected lines represent response (25) and correlation (26) propagators, respectively.

expression (A.2) has a dimensional pole for $\epsilon \to 0$, which is removed once the renormalized quantities (with subscript R) are introduced. In particular (following [3]):

$$\psi_R = Z_\psi^{1/2}\psi, \quad \tilde{\psi}_R = Z_\psi^{1/2}\tilde{\psi}, \quad D_R = Z_D D, \quad \text{and} \quad \tilde{v}_R = Z_{\tilde{v}}\mu^{-\epsilon}\tilde{v}, \quad (A.3)$$

where μ is an arbitrary momentum scale and $Z_\psi = 1 - 4\tilde{v}/\epsilon + O(\tilde{v}^2)$, $Z_D = 1 + 2\tilde{v}/\epsilon + O(\tilde{v}^2)$ and $Z_{\tilde{v}} = 1 - 24\tilde{v}/\epsilon + O(\tilde{v}^2)$ are suitable renormalization constants. Taking into account the definition of σ: $\sigma(t) = (8\pi)^{d/4}D\tilde{v}^{1/2}m(t)$ and that $m_R = Z_\psi^{1/2}m$, one finds for the renormalized function $\sigma_R(t)$ at the fixed-point $\tilde{v}_R^* = \epsilon/24 + O(\epsilon^2)$ [3]

$$\sigma_R(t) = \frac{1}{t}\left(1 + \frac{\epsilon}{4}\ln t - 9\epsilon\right) + O(\epsilon^2), \quad (A.4)$$

which scales according the long-time limit of equation (9) with the proper exponents [see before equation (29) in section 3] and $A_m/B_m = [1 - 9\epsilon + O(\epsilon^2)]/u^*$. To compute the response function to first order in $\tilde{v}$, one has to take into account that the Gaussian propagator gets modified because of the $O(\tilde{v})$-term in equation (A.2) [27]. Using (A.2) in equation (23), one gets (hereafter $t > s$ and $x := s/t$)

$$R_{q=0}^{\prime(0)}(t,s) = x^2\left[1 + \tilde{v}s^{\epsilon/2}\left(\frac{24}{\epsilon}\ln x - 18\ln x - 6\ln^2 x\right) + O(\tilde{v}^2, \epsilon\tilde{v})\right].(A.5)$$

In addition, there are two diagrams contributing to the response function at order $\tilde{v}$, depicted in figure A1. Notice that we can use the $O(\tilde{v}^0)$-propagators to compute them up to $O(\tilde{v})$. We first need the expressions for the following bubbles

$$B_{RC}(t_1, s_1) := \int(\mathrm{d}q)\, R_q^{(0)}(t_1, s_1)C_q^{(0)}(t_1, s_1)$$

$$= 2(8\pi)^{-\frac{d}{2}}t_1^{-4}\int_0^{s_1}\mathrm{d}t'\, t'^3(t_1 - t')^{-\frac{d}{2}} \quad (A.6)$$

$$B_{RR}(t_1, s_1) := \int(\mathrm{d}q)\left[R_q^{(0)}(t_1, s_1)\right]^2 = (8\pi)^{-\frac{d}{2}}\left(\frac{s_1}{t_1}\right)^4(t_1 - s_1)^{-\frac{d}{2}} \quad (A.7)$$

for $t_1 \le s_1$, whereas for $s_1 > t_1$ both B_{RC} and B_{RR} vanish because of causality. The contribution I_1 is then given by the integral

$$I_1 = \int_s^t \mathrm{d}t_1 \int_s^{t_1} \mathrm{d}s_1 \, R_{q=0}^{(0)}(t, t_1) B_{RC}(t_1, s_1) R_{q=0}^{(0)}(s_1, s) \tag{A.8}$$

which has essentially been calculated in [27]. (Indeed, taking into account that $B_{RC}(t_1, s_1)$ here is equal to $t_1^{-1} H_d(t_1, s_1)$ of [27] — see equation (A.4) therein — one finds that the rhs of equation (A.8) is equal to $(s/t)^{1/2} I_2(t, s)/3$ where I_2 is given by equation (A.11) of [27].) The result is

$$I_1 = (8\pi)^{-\frac{d}{2}} x^2 s^{\epsilon/2} \left[-\frac{4}{\epsilon} \ln x - 1 + x + \frac{2\pi^2}{3} + 6\ln(1-x)\left(\frac{1}{x} - 1\right) + 3\ln x \right.$$
$$\left. + \ln^2 x - 4\mathrm{Li}_2(x) + O(\epsilon) \right] . \tag{A.9}$$

The contribution I_2 is instead given by

$$I_2 := \int_s^t \mathrm{d}t_1 \int_s^{t_1} \mathrm{d}s_1 R_{q=0}^{(0)}(t, t_1) B_{RR}(t_1, s_1) R_{q=0}^{(0)}(s_1, s)$$
$$= (8\pi)^{-\frac{d}{2}} x^2 \int_s^t \mathrm{d}t_1 \, t_1^{-2} \int_s^{t_1} \mathrm{d}s_1 \, s_1^2 (t_1 - s_1)^{\epsilon/2-2} . \tag{A.10}$$

We compute first the integral over s_1, which yields after some manipulations and for generic ϵ:

$$\int_s^{t_1} \mathrm{d}s_1 \, s_1^2 (t_1 - s_1)^{\epsilon/2-2} = \frac{(t_1 - s)^{\epsilon/2-1} s^2}{\epsilon/2 + 1} + \frac{2(t_1 - s)^{\epsilon/2-1} t_1 s}{(\epsilon/2 - 1)(\epsilon/2 + 1)} \tag{A.11}$$
$$+ \frac{2(t_1 - s)^{\epsilon/2} t_1}{(\epsilon/2 - 1)\epsilon/2(\epsilon/2 + 1)} .$$

The resulting three integrals over t_1 are easy and one finally gets:

$$I_2 = (8\pi)^{-\frac{d}{2}} x^2 s^{\epsilon/2} \left[-\frac{2}{\epsilon} + \frac{4\ln x}{\epsilon} - \ln^2 x + \frac{\pi^2}{3} \right.$$
$$\left. + 2\mathrm{Li}_2(x) + x - \ln(1-x) - 2 \right] \tag{A.12}$$

where the relation $\mathrm{Li}_2(1 - x^{-1}) = -\ln^2 x/2 - \pi^2/6 + \ln x \ln(1 - x) + \mathrm{Li}_2(x)$ was used. Collecting the different contributions we get the final expression for the response function:

$$R_{q=0}(t, s) = R_{q=0}'^{(0)}(t, s) + 4v I_1 - 2v I_2 + O(v^2, \epsilon v) \tag{A.13}$$
$$= x^2 \left\{ 1 + \tilde{v} s^{\epsilon/2} \left[\frac{4}{\epsilon} - 6\ln x + 2\pi^2 - 12\mathrm{Li}_2(x) \right. \right.$$
$$\left. \left. - 22\ln(1 - x) + 2x + \frac{24\ln(1 - x)}{x} \right] \right\} + O(\tilde{v}^2, \epsilon\tilde{v}) . \tag{A.14}$$

Its dimensional pole is properly removed by introducing the renormalized quantities according to equation (A.3), so that $R_{R,q=0} = Z_\psi R_{q=0}$ which gives indeed the expression in equation (30) for $\tilde{v} = \tilde{v}_R^* = \epsilon/24 + O(\epsilon^2)$.

[1] Harris T E, 1974 *Ann. Probab.* **2** 969
[2] Janssen H K and Täuber U C, 2005 *Ann. Phys.* **315** 147
[3] Täuber U C, Howard M H and Vollmayr-Lee B P, 2005 *J. Phys. A: Math. Gen.* **38** R79
    Täuber U C, 2006 in *Ageing and the glass transition* ed M Henkel, M Pleimling and R Santuary, Springer Lecture Notes in Physics (Heidelberg: Springer) [cond-mat/0511743]
[4] Hinrichsen H, 2000 *Adv. Phys.* **49** 815 and references therein
[5] Ódor G, 2004 *Rev. Mod. Phys.* **76** 663
[6] Janssen H K, 1981 *Z. Phys. B* **42** 151
[7] Cardy J L and Sugar R L, 1980 *J. Phys. A: Math. Gen.* **13** L423
[8] Enss T, Henkel M, Picone A and Schollwöck U, 2004 *J. Phys. A: Math. Gen.* **37** 10479
[9] Ramasco J J, Henkel M, Santos M A and da Silva Santos C A, 2004 *J. Phys. A: Math. Gen.* **37** 10497
[10] Hinrichsen H, 2006 *J. Stat. Mech.* L06001
[11] Henkel M, Enss T and Pleimling M, 2006 *J. Phys. A: Math. Gen.* **39** L589 [cond-mat/0605211]
[12] Godrèche C and Luck J M, 2000 *J. Phys. A: Math. Gen.* **33** 9141 [cond-mat/0001264]
     Godrèche C and Luck J M, 2002 *J. Phys.: Condens. Matter* **14** 1589 [cond-mat/0109212]
[13] Calabrese P and Gambassi A, 2005 *J. Phys. A: Math. Gen.* **38** R133 [cond-mat/0410357]
[14] Cugliandolo L F and Kurchan J, 1993 *Phys. Rev. Lett.* **71** 173 [cond-mat/9303036]
     Cugliandolo L F and Kurchan J, 1994 *J. Phys. A: Math. Gen.* **27** 5749 [cond-mat/9311016]
     Cugliandolo L F and Kurchan J, 1995 *Phil. Mag. B* **71** 50 [cond-mat/9403040]
     Cugliandolo L F, Kurchan J and Parisi G, 1994 *J. Phys. I (France)* **4** 1641 [cond-mat/9406053]
[15] Henkel M, 2002 *Nucl. Phys.* **B641** 405
[16] Henkel M, Picone A and Pleimling M, 2004 *Europhys. Lett.* **68** 191
[17] Picone A and Henkel M, 2004 *Nucl. Phys.* **B688** 217
[18] Baumann F, Stoimenov S and Henkel M, 2006 *J. Phys. A: Math. Gen.* **39** 4095
[19] Ódor G, 2006 cond-mat/0606724
[20] Calabrese P and Gambassi A, 2002 *Phys. Rev. E* **66** 066101
[21] Pleimling M and Gambassi A, 2005 *Phys. Rev. B* **71** 180401(R)
[22] Lippiello E, Corberi F and Zannetti M, 2006 cond-mat/0604321
[23] Henkel M, 2006 cond-mat/0609672
[24] Zinn-Justin J 1996, *Quantum Field Theory and Critical Phenomena* 3rd edition (Oxford: Clarendon)
[25] Oerding K and van Wijland F, 1998 *J. Phys. A: Math. Gen.* **31** 7011
[26] Privman V, Hohenberg P C and Aharony A, 1991 in *Phase Transitions and Critical Phenomena* vol 14 ed C Domb and J L Lebowitz (New York: Academic)
[27] Calabrese P, Gambassi A and Krzakala F, 2006 *J. Stat. Mech.* P06016
[28] van Wijland F, Oerding K and Hilhorst H J, 1998 *Physica A* **251** 179
[29] Janssen H K, Schaub B and Schmittmann B, 1989 *Z. Phys. B* **73** 539
     Janssen H K, 1992 in *From Phase Transitions to Chaos– Topics in Modern Statistical Physics* ed G Györgyi, I Kondor, L Sasvári and T Tel (Singapore: World Scientific)
[30] Calabrese P and Gambassi A 2002 *Phys. Rev. E* **65** 066120 [cond-mat/0203096]
     Calabrese P and Gambassi A 2002 *Acta Phys. Slov.* **52** 311
     Calabrese P and Gambassi A 2002 *Phys. Rev. B* **66** 212407 [cond-mat/0207487]
     Gambassi A 2006 *J. Phys.: Conf. Series* **40** 13
[31] Cugliandolo L F, 2003 in *Slow relaxation and nonequilibrium dynamics in condensed matter* Les Houches, École d'Été de Physique Théorique vol 77 ed J-L Barrat *et al* (Berlin: Springer) p 371 [cond-mat/0210312]
[32] Enss T, 2006 private communication